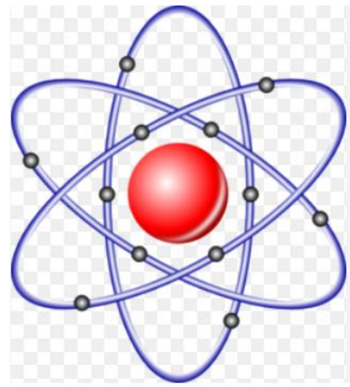
THE FIVE LAWS OF EXPONENTS



Your chemistry teacher asks you to find the total mass of 3.01×10^{23} electrons assuming that the mass of one election is 9.1×10^{-31} kg. To solve this problem, you need to multiply the two numbers (which is explained in the chapter, *Scientific Notation*). Multiplying 3.01×9.1 is easy; just multiply the decimals. But how do we multiply $10^{23} \times 10^{-31}$? *What do we do with those exponents?* This chapter will answer that question.

❑ *EXPONENTS WITH NUMBERS*

Let's begin by avoiding variables for the moment and just stick to numbers so we can experiment with combining expressions containing exponents.

I. We want a way to simply an expression like

$$2^3 \times 2^4$$

Maybe we multiply the 2's and multiply the 3 and the 4. Perhaps we don't multiply the 2's but still multiply the 3 and the 4. Or maybe we need to *add* the 3 and the 4. It's hard to know what to do unless you already know what to do! But we can figure out what to do if we just do the arithmetic:

$$2^3 \times 2^4 \;=\; 8 \times 16 \;=\; 128 \;=\; \mathbf{2^7}$$

From this example, it appears we simply keep the base (the 2) and *add* the exponents. Here's another example where this works:

$$3^2 \cdot 3^3 \;=\; 9 \cdot 27 \;=\; 243 \;=\; \mathbf{3^5}$$

I think we have a neat rule here: *To multiply powers of the same base, just keep the base and <u>add</u> the exponents*. This powerful rule means we can multiply powers <u>without</u> doing any heavy arithmetic; for example:

$$12^9 \times 12^{41} \;=\; \mathbf{12^{50}}$$ [Try <u>that</u> the long way!]

> **Important Note:** This shortcut is for <u>multiplying</u> powers of a base; it does <u>not</u> work if the operation is addition. Check this:
>
> $$2^2 + 2^3 \;=\; 4 + 8 \;=\; 12$$
>
> But 12 is <u>not</u> a nice power of 2 (since 2^3 is 8 and 2^4 is 16).
> So if you want to add exponents, be sure that the bases are the same and that the operation is multiplication.

II. Now we study what to do with a "power of a power." For example, we might come across $\left(2^2\right)^3$. Do we add the exponents, or maybe multiply them? Let's just work it out and see:

$$\left(2^2\right)^3 \;=\; 4^3 \;=\; 64 \;=\; 2^6$$

It appears we simply multiply the exponents. This means we can simplify something like $\left(5^7\right)^{10}$ by just raising 5 to the product of 7 and 10:

$$\left(5^7\right)^{10} \;=\; 5^{70}\text{, and we're done!}$$

III.

❑ EXPONENTS WITH VARIABLES

For each of the following five examples, we will "stretch-and-squish," and then we'll generalize what we observe to the official ***Five Laws of Exponents***.

I. We start by finding the product of x^3 and x^4:

$$x^3x^4 = (x\,x\,x)(x\,x\,x\,x) = x\,x\,x\,x\,x\,x\,x = x^7$$

Notice that the bases (the x's) are the same, and it's a multiplication problem. As long as the bases are the same, and it's a multiplication problem, it appears that we merely need to write down the base, and then <u>add</u> the exponents together to get the exponent of the answer. That is, $x^ax^b = x^{a+b}$.

$$\boxed{x^3x^4 = x^7}$$

[This law of exponents, where we multiply powers of the same base by <u>adding</u> the exponents, is almost always called the First Law of Exponents. The four laws that follow are not in any particular order.]

II. For our second example, let's raise a power to a power:

$$(x^4)^2 = (x\,x\,x\,x)^2 = (x\,x\,x\,x)(x\,x\,x\,x) = x\,x\,x\,x\,x\,x\,x\,x = x^8$$

We appear to have a shortcut at hand. Simply <u>multiply</u> the two exponents together and we're done. So, to raise a power to a power, we can write a general rule: $(x^a)^b = x^{ab}$.

$$\boxed{(x^4)^2 = x^8}$$

III. Now we're to try raising a product to a power; for instance,

$$(ab)^5 = (ab)(ab)(ab)(ab)(ab) = (aaaaa)(bbbbb) = a^5b^5$$

In general, when raising a product to a power, <u>raise each factor</u> to the power: $(xy)^n = x^n y^n$.

$$\boxed{(ab)^5 = a^5b^5}$$

Note that the quantity in the parentheses is a <u>single</u> term -- there's no adding or subtracting in the parentheses. In fact, if there are two or more terms in the parentheses, this law of exponents does <u>not</u> apply.

IV. Next we divide powers of the same base. We'll need two examples for this law of exponents.

A. $\dfrac{x^6}{x^2} = \dfrac{xxxxxx}{xx} = \dfrac{\cancel{x}\cancel{x}xxxx}{\cancel{x}\cancel{x}} = x^4$

B. $\dfrac{y^4}{y^6} = \dfrac{yyyy}{yyyyyy} = \dfrac{\cancel{y}\cancel{y}\cancel{y}\cancel{y}}{\cancel{y}\cancel{y}\cancel{y}\cancel{y}yy} = \dfrac{1}{y^2}$

$$\boxed{\dfrac{x^6}{x^2} = x^4 \qquad \dfrac{y^4}{y^6} = \dfrac{1}{y^2}}$$

In general, when dividing powers of the same base, <u>subtract</u> the exponents, leaving the remaining factors on the top if the top exponent is bigger, and on the bottom if the bottom exponent is bigger.

V. Our last example in this section is the process of raising a quotient to a power. As usual, we stretch and squish; then we generalize to a law of exponents.

$$\left(\frac{a}{b}\right)^4 = \left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right) = \frac{aaaa}{bbbb} = \frac{a^4}{b^4}$$

$$\boxed{\left(\frac{a}{b}\right)^4 = \frac{a^4}{b^4}}$$

In general, we can raise a quotient to a power by raising both the top and bottom to the power: $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

❑ *SUMMARY OF THE FIVE LAWS OF EXPONENTS*

Exponent Law	Example
$x^a x^b = x^{a+b}$	$x^2 x^6 = x^8$
$(x^a)^b = x^{ab}$	$(a^4)^3 = a^{12}$
$(xy)^n = x^n y^n$	$(wz)^7 = w^7 z^7$
For $a > b$, $\dfrac{x^a}{x^b} = x^{a-b}$ For $b > a$, $\dfrac{x^a}{x^b} = \dfrac{1}{x^{b-a}}$	$\dfrac{x^{10}}{x^2} = x^8$ $\dfrac{a^3}{a^7} = \dfrac{1}{a^4}$
$\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}$	$\left(\dfrac{a}{b}\right)^5 = \dfrac{a^5}{b^5}$

❑ *SINGLE-STEP EXAMPLES*

EXAMPLE 1 [Let's call this law the *First Law of Exponents*.]

A. $A^7 A^5 = A^{7+5} = A^{12}$

The bases are the same, and it's a multiplication problem. So we can simply write the base and add the exponents.

B. $x^2x^3x^4 = x^{2+3+4} = x^9$

All the bases are the same, and it's a multiplication problem, and so we simply add the exponents.

C. $(x+y)^4(x+y)^9 = (x+y)^{13}$

It doesn't matter what the base is, as long as we're multiplying powers of the <u>same</u> base.

EXAMPLE 2:

A. $(c^{10})^2 = c^{20}$

Raising a base to a power, and then raising that result to a further power requires simply that we multiply the exponents.

B. $\left((x^2)^3\right)^4 = x^{24}$

$(x^2)^3 = x^6$, and then $(x^6)^4 = x^{24}$
Shortcut: Just multiply all three exponents.

EXAMPLE 3:

A. $(ax)^5 = a^5x^5$

It's a power of a product (a single term). So just raise each factor to the 5th power.

B. $(abc)^7 = a^7b^7c^7$

Even a term with three factors can be raised to the 7th power by raising each factor to the 7th power.

EXAMPLE 4:

A. $\dfrac{x^7}{x^5} = x^2$

Since 7 > 5, we divide powers of the same base by subtracting the exponents.

B. $\dfrac{w^{15}}{w^{25}} = \dfrac{1}{w^{10}}$

Since the bigger exponent is on the bottom, we subtract 15 from 25 and leave that power of w on the bottom.

EXAMPLE 5:

A. $\left[\dfrac{x}{z}\right]^7 = \dfrac{x^7}{z^7}$

To raise a quotient to a power, just raise both the top and bottom to the 7th power.

B. $\left(\dfrac{a+b}{u-w}\right)^{23} = \dfrac{(a+b)^{23}}{(u-w)^{23}}$

Just raise top and bottom to the 23rd power.

Homework

1. Use the Five Laws of Exponents to simplify each expression:

a. a^3a^4 b. $x^5x^6x^2$ c. y^3y^3 d. $z^{12}z$

e. $(x^3)^4$ f. $(z^8)^2$ g. $(n^{10})^{10}$ h. $(a^1)^7$

i. $(ab)^3$ j. $(xyz)^5$ k. $(RT)^1$ l. $(math)^5$

m. $\dfrac{a^8}{a^2}$ n. $\dfrac{b^3}{b^9}$ o. $\dfrac{w^5}{w^5}$ p. $\dfrac{Q^{100}}{Q^{50}}$

q. $\left(\dfrac{k}{w}\right)^4$ r. $\left(\dfrac{a}{b}\right)^{99}$ s. $\left(\dfrac{1}{m}\right)^{20}$ t. $a(bc)^2$

2. Use the Five Laws of Exponents to simplify each expression:

a. a^3a^5 b. $u^5u^7u^2$ c. $y^{30}y^{30}$ d. $z^{14}z$

e. $(x^4)^5$ f. $(z^9)^2$ g. $(n^{100})^{10}$ h. $(a^1)^9$

i. $(xy)^4$ j. $(abc)^{17}$ k. $(pn)^1$ l. $(love)^4$

m. $\dfrac{a^{10}}{a^2}$ n. $\dfrac{b^3}{b^{12}}$ o. $\dfrac{w^9}{w^9}$ p. $\dfrac{Q^{100}}{Q^{20}}$

q. $\left(\dfrac{x}{w}\right)^3$ r. $\left(\dfrac{a}{b}\right)^{999}$ s. $\left(\dfrac{1}{z}\right)^{22}$ t. $w(xy)^3$

❑ *WHEN NOT TO USE THE FIVE LAWS OF EXPONENTS*

$\boldsymbol{a^5b^6}$ cannot be simplified. Although the First Law of Exponents demands that the expressions be multiplied — and they are — it also requires that the bases be the same — and they aren't.

$\boldsymbol{x^3 + x^4}$ cannot be simplified. Even though the bases are the same, the First Law of Exponents requires that the two powers of x be <u>multiplied</u>.

$\boldsymbol{w^3 + w^3}$ can be simplified, but not by the First Law of Exponents, since the powers of w are not being multiplied. But the two terms are *like terms*, which means we simply add them together to get $\boldsymbol{2w^3}$.

$\boldsymbol{(a + b)^{23}}$ does not equal $a^{23} + b^{23}$. You may think that the third law of exponents, $(xy)^n = x^n y^n$, might apply, but it does not, and that's because xy is a single term, whereas $a + b$ consists of two terms. You'll have to wait until the chapter entitled *The Binomial Theorem* to learn a clever way to calculate the 23rd power of $a + b$. Also, you may have already learned in this class that $(a + b)^2$ is actually equal to $a^2 + 2ab + b^2$, and so again, $(a + b)^n \neq a^n + b^n$ (for $n \geq 2$).

Homework

3. Simplify each expression:

a. y^4y^4	b. a^3b^4	c. $x^4x^3x^2$	d. $p^3t^2p^2$
e. $a^3 + a^5$	f. a^3a^5	g. $n^4 + n^4$	h. $x^3 - x^3$
i. $(x + y)^{55}$	j. $Q^2 + Q^2$	k. u^5w^6	l. $h^6 - h^2$
m. $(a - b)^2$	n. $(ab)^2$	o. $(x^3)^3$	p. $x^4 + x^5$
q. $x^{14} + x^{14}$	r. $y^{12} - y^{12}$	s. $a^8 + a^9$	t. $a^{10} + a^{10}$
u. $(xy)^2$	v. $(x + y)^2$	w. a^3b^4	x. $a^3 + b^4$
y. $a(a^2)(b^2)b$	z. $n^6 + n^6$		

❑ *MULTI-STEP EXAMPLES*

EXAMPLE 6:

A. $(-3x^2y^3)(-5xy^7) = (-3)(-5)\,(x^2x)(y^3y^7) = 15x^3y^{10}$

B. $-2x^2y(xy - 4x^3y^4) = -2x^3y^2 + 8x^5y^5$ (distribute)

C. $\left(2a^2b^3\right)^4 = 2^4(a^2)^4(b^3)^4 = 16a^8b^{12}$ (the 2 <u>is</u> in the parentheses)

D. $7\left(xy^{10}\right)^5 = 7x^5(y^{10})^5 = 7x^5y^{50}$ (the 7 is <u>not</u> in the parentheses)

E. $\left(\dfrac{a^2}{b^3}\right)^7 = \dfrac{(a^2)^7}{(b^3)^7} = \dfrac{a^{14}}{b^{21}}$

F. $\left(\dfrac{x^3y^9}{xy^{12}}\right)^5 = \left(\dfrac{x^2}{y^3}\right)^5 = \dfrac{(x^2)^5}{(y^3)^5} = \dfrac{x^{10}}{y^{15}}$ (simplify the inside first)

Homework

4. Simplify each expression:

a. $(-5a^3b^4)(-2a^2b)$

b. $(7xy)(-7x^2y^5)$

c. $(-2uw)(2uw)$

d. $x^3(2x^2 - x - 1)$

e. $3y^2(3y^2 - y + 3)$

f. $\left(a^2b^3\right)^4$

g. $\left(-5m^3n^{10}\right)^3$

h. $\left[-3p^3q^3\right]^4$

i. $4\left(xy^7\right)^{10}$

j. $-10(-2c^3y^4)^3$

k. $\left(\dfrac{a^3}{c^2}\right)^{10}$

l. $\left[\dfrac{2x^3}{3xy^4}\right]^3$

m. $\left(\dfrac{a^2b^3}{a^4b}\right)^5$ n. $2(3x^2y^3)^4$

❑ ZERO AS AN EXPONENT

We've already learned that anything to the zero power is 1 (as long as it's not zero to the zero). We reached this conclusion after we learned that $2^0 = 1$, and figured it might be true for any base. Now we try to verify this; that is, what is $\boldsymbol{x^0}$? Consider the expression

x^3x^0 where we assume $x \neq 0$.

To figure out the meaning of x^0, we can use the First Law of Exponents to calculate

$$x^3x^0 = x^{3+0} = x^3$$

That is,

$$x^3x^0 = x^3$$

Now "isolate" the x^0, since that's what we're trying to find the value of. We do this by dividing each side of the equation by x^3:

$$\frac{x^3x^0}{x^3} = \frac{x^3}{x^3},$$

It's legal to divide by x^3, since we've stipulated that $x \neq 0$.

which implies that

$$\boldsymbol{x^0 = 1},$$

and we're done:

> **Any number (except 0) raised to the zero power is 1.**

<u>EXAMPLE 7:</u>

A. $(x - 3y + z)^0 = \mathbf{1}$ (any quantity ($\neq 0$) to the zero power is 1)

B. $(abc)^0 = \mathbf{1}$ (any quantity ($\neq 0$) to the zero power is 1)

C. $a + b^0 = \boldsymbol{a + 1}$ (the exponent is on the b only)

D. $uw^0 = u(1) = \boldsymbol{u}$ (the exponent is on the w only)

E. $(-187)^0 = \mathbf{1}$ (the exponent is on the -187)

F. $-14^0 = \mathbf{-1}$ (the exponent in on the 14, not on the minus sign)

Homework

5. Evaluate each expression:

a. $2^0 + 3^0$ b. $4^0 \cdot 5^0$ c. $6^0 + 2^1 + 2^2 + 2^3 + 2^4$

d. $(1 + 12)^0$ e. $2^5 - 2^3 + 2^1 - 2^0$ f. $(8 - 5)^0 + (10 - 8)^1$

g. $2^0 \times 2^1 \times 2^2 \times 2^3 \times 2^4$

h. $\left(\frac{12}{29}\right)^0 + \left(\frac{100}{77}\right)^0 - (20 - \pi - 3)^0 + \left(3^2 - 7\right)^1$

6. Simplify each expression:

a. x^0 b. xy^0 c. $x + y^0$ d. $(x + y)^0$

e. $(ab)^0$ f. $\left(\frac{a^2}{b^3}\right)^0$ g. $\frac{(x^2)^0}{y^3}$ h. m^0m

i. $x^0 + x^0$ j. Q^0Q^0 k. $a^0 - a^0$ l. $\left(\frac{2x^2y^0}{-3ab^{10}}\right)^0$

Review Problems

7. Simplify: $\left(-3x^3y^4x^7\right)^3$

8. Simplify: $-3\left(x^5x^4x^8\right)^3$

9. Simplify: $\dfrac{a^2b^3c^9}{ab^4c^3}$

10. Simplify: $-2y^3(3y^4 - 2y^3 + 1)$

11. Simplify: $x^{12} + x^{14}$

12. Simplify: $u^{22} + u^{22}$

13. Simplify: $abcd^0e^0$

14. Simplify: $\left[\dfrac{10^0a^0b^{14}}{a^3b^7}\right]^5$

Solutions

1.

a. a^7	b. x^{13}	c. y^6	d. z^{13}
e. x^{12}	f. z^{16}	g. n^{100}	h. a^7
i. a^3b^3	j. $x^5y^5z^5$	k. RT	l. $m^5a^5t^5h^5$
m. a^6	n. $\dfrac{1}{b^6}$	o. 1	p. Q^{50}
q. $\dfrac{k^4}{w^4}$	r. $\dfrac{a^{99}}{b^{99}}$	s. $\dfrac{1}{m^{20}}$	t. ab^2c^2

2.

a. a^8	b. u^{14}	c. y^{60}	d. z^{15}
e. x^{20}	f. z^{18}	g. n^{1000}	h. a^9
i. x^4y^4	j. $a^{17}b^{17}c^{17}$	k. pn	l. $l^4o^4v^4e^4$
m. a^8	n. $\dfrac{1}{b^9}$	o. 1	p. Q^{80}

q. $\frac{x^3}{w^3}$ r. $\frac{a^{999}}{b^{999}}$ s. $\frac{1}{z^{22}}$ t. wx^3y^3

3. a. y^8 b. As is c. x^9 d. p^5t^2
e. As is f. a^8 g. $2n^4$ h. 0
i. As is (for now) j. $2Q^2$ k. As is l. As is
m. $a^2 - 2ab + b^2$ n. a^2b^2 o. x^9 p. As is
q. $2x^{14}$ r. 0 s. As is t. $2a^{10}$
u. x^2y^2 v. $x^2 + 2xy + y^2$ w. As is
x. As is y. a^3b^3 z. $2n^6$

4. a. $10a^5b^5$ b. $-49x^3y^6$ c. $-4u^2w^2$ d. $2x^5 - x^4 - x^3$
e. $9y^4 - 3y^3 + 9y^2$ f. a^8b^{12} g. $-125m^9n^{30}$
h. $81p^{12}q^{12}$ i. $4x^{10}y^{70}$ j. $80c^9y^{12}$ k. $\frac{a^{30}}{c^{20}}$
l. $\frac{8x^6}{27y^{12}}$ m. $\frac{b^{10}}{a^{10}}$ n. $162x^8y^{12}$

5. a. 2 b. 1 c. 31 d. 1
e. 25 f. 3 g. 1024 h. 3

6. a. 1 b. x c. $x + 1$ d. 1 e. 1 f. 1
g. $\frac{1}{y^3}$ h. m i. 2 j. 1 k. 0 l. 1

7. $-27x^{30}y^{12}$ **8.** $-3x^{51}$ **9.** $\frac{ac^6}{b}$

10. $-6y^7 + 4y^6 - 2y^3$ **11.** As is **12.** $2u^{22}$

13. abc **14.** $\frac{b^{35}}{a^{15}}$

I have no particular talent.
I am merely inquisitive.

Albert Einstein